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Technical Notes

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Damped-Wave Conduction and Relaxation in a Finite Sphere and Cylinder

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I. Introduction

SIX reasons were given by Sharma [1] to seek a generalized Fourier's Law of Heat Conduction. Boley [2] showed that the addition of the second derivative in time of temperature to the governing equation is the only mathematical way to remove singularities found in the solution to parabolic heat conduction equations. When the relaxation time is zero, the generalized Fourier's law of heat conduction will revert to the parabolic partial differential equation (PDE) for transient conduction from Fourier's law. When the rate of temperature change over time is much greater than an exponential rise with time, e^{τ}, the generalized Fourier's law of heat conduction equation will revert to the wave equation (Tzou [3] and Sharma [4]). Eq. (1) is a hyperbolic partial differential equation, that is second order with respect to space and second order with respect to time. Reference to the generalized Fourier's law of heat conduction can be traced back to Maxwell [5], Morse and Feshbach [6]. Cattaneo [7] and Vernotte [8] postulated the generalized Fourier's law of heat conduction and relaxation equation independently. This equation can be used to account for the finite speed of heat. Reviews in the literature of the use of Eq. (1) have been provided by Joseph and Preziosi [9] and Ozisik and Tzou [10]. Sharma [1,4,11–13] discussed the manifestation of the damped-wave transport and relaxation equation in industrial applications and provided bounded solutions within the constraints of the second law of thermodynamics. It was shown that the generalized Fourier's law of heat conduction can be derived by including the acceleration term in the free-electron theory, the acceleration term in the Stokes–Einstein theory for molecular diffusion, by accounting for the accumulation term in the kinetic theory of gases and combining, in series, Hooke's elastic element and Newton's viscous element in the viscoelastic theory. The relaxation time was found to be one-third of the collision time of the electron and the obstacle. The velocity of heat was found to be identical with the velocity of mass derived from kinetic representation of pressure or the Maxwell representation of the speed of molecules. They derive a set of equations for length scales comparable with the mean free path of the molecule. Ali [14,15] used statistical mechanics and kinetic theory and derived the generalized Fourier's law of heat conduction for monatomic and diatomic gases. Glass and McRae [16] looked at the variable specific heat and thermal relaxation parameter. Mitra et al. [17] presented experimental evidence of the wave nature of heat propagation in processed meat and demonstrated that the hyperbolic heat conduction model is an accurate representation on a macroscopic level of the heat conduction process in such biological material. They report a relaxation time of the order of 16 s.

Some investigators have raised some concerns about violations of the second law of thermodynamics by the generalized Fourier's law of heat conduction (Bai and Lavine [18], Taitel [19], Zanchini [20], and Barletta and Zanchini [21]).

Sharma [4] presented an analytical solution for the case of a finite slab subject to constant wall temperature. The final condition in time as the fourth condition for the second-order hyperbolic PDE governing equation was shown to result in well-bounded solutions. This means that care must go into the choice of the conditions used in the boundaries of space and the initial and final time values. They have to be physically realistic such that equilibrium temperature is attained at steady state. Only for large relaxation times were oscillations found in the solution for temperature. These oscillations were found to be subcritical and damped. The time conditions used by Taitel [19] are unrealistic from the physical realities of heat transfer. That is why his solution exhibited a temperature overshoot. Thus, the equations do not violate the laws of thermodynamics as much as the choice of space and time conditions as constraints. Sharma [13] also showed that a temperature undershoot can occur when Fourier's law of heat conduction is applied at steady state in the presence of a temperature-dependent heat source. This is in violation of the third law of thermodynamics. Here again, the choice of the space condition at some arbitrary length is not sufficient. The critical length beyond which no heat transfer will occur will have to be identified to keep the solution from violating the third law of thermodynamics.

Very little work is reported in the literature on the analytical solutions to the damped-wave conduction and relaxation equation for the cases of a finite sphere and a finite cylinder. Jiang [22] reported a solution to the hyperbolic heat equation in hollow spherical objects by the method of integral transformation. Tsai and Hung [23] studied the dynamic behavior of a bilayered composite sphere subject to sudden change on the outer surface using the Riemann-sum approximation and Laplace transformation. In this study, analytical solutions are developed for the case of a finite sphere and a finite cylinder subject to the constant-wall-temperature boundary condition by the method of separation of variables. The use of the final condition in time is made in place of the time derivative of temperature at zero time to obtain well-bounded solutions. The steady-state wave temperature has to reach equilibrium. Allowing one component of the solution, the wave temperature, to grow exponentially with time may be why the previous investigators found a temperature overshoot under certain conditions.

II. Finite Sphere Subject to Constant Wall Temperature

Consider a sphere at initial temperature T_0. The surface of the sphere is maintained at a constant temperature T_s for times greater than zero. The heat propagative velocity is given as the square root of

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the ratio of thermal diffusivity and relaxation time, $V_h = \sqrt{\alpha/\tau_r}$. The governing equation and the initial, final, and boundary conditions are as follows:

$$\tau_r \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} \tag{1}$$

$$t = 0, \qquad 0 \le r < R, \qquad T = T_0 \tag{2}$$

$$t = \infty, \qquad 0 \le r < R, \qquad T = T_s \tag{3}$$

$$t > 0, \qquad r = 0, \qquad \partial T/\partial r = 0 \tag{4}$$

$$t > 0, \qquad r = R, \qquad T = T_s \tag{5}$$

The governing equation can be obtained by eliminating q_r between the generalized Fourier's law of heat conduction and the equation from the energy balance of in–out accumulation. This is achieved by differentiating the constitutive equation with respect to r and the energy equation with respect to t and eliminating the second cross derivative of q with respect to r and time. In the dimensionless form, Eq. (1) can be written in cylindrical coordinates as

$$\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial X^2} + \frac{2}{X}\frac{\partial u}{\partial X} \tag{6}$$

where

$$X = \frac{r}{\sqrt{\alpha\tau_r}}, \qquad \tau = \frac{t}{\tau_r}, \qquad u = \frac{(T - T_s)}{(T_0 - T_s)} \tag{7}$$

The solution is obtained by the method of separation of variables. Firs, t the damping term is removed by the substitution $u = e^{-\tau/2}w$. By this substitution, Eq. (7) becomes

$$-\frac{w}{4} + \frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial X^2} + \frac{2}{X}\frac{\partial w}{\partial X} \tag{8}$$

The method of separation of variables can be used to obtain the solution of Eq. (8). Let

$$w = V(\tau)\phi(X) \tag{9}$$

Substituting Eq. (9) into Eq. (8) and separating the variables that are a function of X only and τ only, the following two ordinary differential equations, one in space and another in time, are obtained:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}X^2} + \frac{2}{X}\frac{\mathrm{d}\phi}{\mathrm{d}X} + \lambda^2\phi = 0 \tag{10}$$

$$\frac{\mathrm{d}^2 V}{\mathrm{d}\tau^2} = \left(\frac{1}{4} - \lambda^2\right) = 0 \tag{11}$$

The solution for Eq. (10) is the Bessel function of one-half order and the first kind:

$$\phi = c_1 J_{1/2}(\lambda X) + c_2 J_{-1/2}(\lambda X) \tag{12}$$

It can be seen that $c_2 = 0$ because the concentration is finite at $X = 0$. From the boundary condition at the surface,

$$\phi = c_1 J_{1/2}\left(\frac{\lambda R}{\sqrt{\alpha\tau_r}}\right) + c_2 J_{-1/2}\left(\frac{\lambda R}{\sqrt{\alpha\tau_r}}\right) \tag{13}$$

$$\frac{\lambda_n R}{\sqrt{\alpha\tau_r}} = (n-1)\pi \quad \text{for } n = 2, 3, 4, \ldots \tag{14}$$

The solution for Eq. (11) is the sum of two exponentials in time: one that decays with time and another that grows exponentially with time.

$$V = c_3 \exp\left(\tau\sqrt{0.25 - \lambda_n^2}\right) + c_4 \exp\left(-\tau\sqrt{0.25 - \lambda_n^2}\right) \tag{15}$$

The term containing the positive exponential power exponent will drop out, because with increasing time, the system may be assumed to reach steady state, and the points within the sphere will always have temperature values less than those at the boundary. From the final condition in time (i.e., at steady state),

$$w = ue^{\tau/2} \tag{16}$$

where w will have to be zero at infinite time, as can be seen from Eq. (16). Thus, c_3 in Eq. (15) is found to be zero. Hence,

$$V = c_4 \exp\left(-\tau\sqrt{0.25 - \lambda_n^2}\right) \tag{17}$$

or

$$u = \sum_0^\infty c_n J_{1/2}(\lambda_n X) \exp\left(\frac{-\tau}{2} - \tau\sqrt{0.25 - \lambda_n^2}\right) \tag{18}$$

The c_n can be solved from the initial condition by using the principle of orthogonality for Bessel functions. At time zero, the left-hand side (LHS) and right-hand side (RHS) are multiplied by $J_{1/2}(\lambda_m X)$. Integration between the limits of 0 and R is performed. When n is not m, the integral is zero from the principle of orthogonality. Thus, when $n = m$,

$$c_n = \frac{-\int_0^R J_{1/2}(\lambda_n X)}{\int_0^R J_{1/2}^2(c\lambda_n X)} \tag{19}$$

It can be noted from Eq. (18) that when

$$\tfrac{1}{4} < \lambda_n^2 \tag{20}$$

the solution will be periodic with respect to the time domain. This can be obtained by using the de Moivre's theorem and obtaining the real part to $\exp(-i\tau\sqrt{\lambda_n^2 - 0.25})$. Thus, for materials with relaxation times greater than a certain limiting value, the solution for temperature will exhibit subcritical damped oscillations. Hence,

$$\tau_r > \frac{R^2}{12.57\alpha} \tag{21}$$

Thus, a bifurcated solution is obtained. From Eq. (19) it can also be seen that all terms in the infinite series will be periodic [i.e., even for $n = 2$ when Eq. (22) is valid]:

$$u = \sum_0^\infty c_n J_{1/2}(\lambda_n X) \cos\left(\tau\sqrt{\lambda_n^2 - 0.25}\right) \tag{22}$$

Thus, the transient-temperature profile in a sphere is obtained for a step change in temperature at the surface of the sphere using the modified Fourier's heat conduction law. For materials with relaxation times greater than $R^2/12.57\alpha$, subcritical damped oscillations can be seen in the transient-temperature profile. The exact solution for transient-temperature profile using finite-speed heat conduction is derived by the method of separation of variables. It is a bifurcated solution. For certain values of λ, the time portion of the solution is "cosinous" and damped and for others it is an infinite series of Bessel functions of the first kind and one-half order and decaying exponential in time. It can also be shown that for terms in the infinite series with n greater than 2, the contribution to the solution will be periodic for small R. The exact solution is bifurcated.

III. Finite Cylinder Subject to Constant Wall Temperature

Consider a cylinder at initial temperature T_0. The surface of the sphere is maintained at a constant temperature T_s for times greater than zero. The heat propagative velocity is given as the square root of the ratio of thermal diffusivity and relaxation time, $V_h = \sqrt{\alpha/\tau_r}$. The initial, final, and boundary conditions are the same as those given for the sphere in Eqs. (2–5). The governing equation can be obtained by eliminating q_r between the generalized Fourier's law of heat conduction and the equation from the energy balance of in–out accumulation. This is achieved by differentiating the constitutive equation with respect to r and the energy equation with respect to t and eliminating the second cross derivative of q with respect to r and time. Thus,

$$\tau_r \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial r^2} + \frac{\alpha}{r}\frac{\partial T}{\partial r} \tag{23}$$

The governing equation in the dimensionless form is then

$$\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial X^2} + \frac{1}{X}\frac{\partial u}{\partial X} \tag{24}$$

The solution is obtained by the method of separation of variables. First, the damping term is removed by the substitution $u = e^{-\tau/2} w$.

$$-\frac{w}{4} + \frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial X^2} + \frac{1}{X}\frac{\partial w}{\partial X} \tag{25}$$

The method of separation of variables can be used to obtain the solution of Eq. (25). Let

$$w = V(\tau)\phi(X) \tag{26}$$

Substituting Eq. (26) into Eq. (25) and separating the variables that are a function of X only and τ only, the following two ordinary differential equations, one in space and another in time, are obtained:

$$\frac{d^2\phi}{dX^2} + \frac{1}{X}\frac{d\phi}{dX} + \lambda^2 \phi = 0 \tag{27}$$

$$\frac{d^2 V}{d\tau^2} = \left(\frac{1}{4} - \lambda^2\right) = 0 \tag{28}$$

The solution to Eq. (27) can be seen to be a Bessel function of the zeroth order and first kind and a Bessel function of the zeroth order and second kind:

$$\phi = c_1 J_0\left(\frac{\lambda R}{\sqrt{\alpha\tau_r}}\right) + c_2 Y_0\left(\frac{\lambda R}{\sqrt{\alpha\tau_r}}\right) \tag{29}$$

It can be seen that $c_2 = 0$ because the temperature is finite at $X = 0$. From the boundary condition considered at the surface,

$$\frac{\lambda_n R}{\sqrt{\alpha\tau_r}} = 2.4048 + (n-1)\pi \quad \text{for } n = 2, 3, 4, \ldots \tag{30}$$

The solution for Eq. (28) is the sum of two exponentials in time: one that decays with time and another that grows exponentially with time.

$$V = c_3 \exp\left(\tau\sqrt{0.25 - \lambda_n^2}\right) + c_4 \exp\left(-\tau\sqrt{0.25 - \lambda_n^2}\right) \tag{31}$$

The term containing the positive exponential power exponent will drop out, because with increasing time, the system may be assumed to reach steady state, and the points within the sphere will always have temperature values less than those at the boundary. From the final condition in time (i.e., at steady state),

$$w = ue^{\tau/2} \tag{32}$$

where w will have to be zero at infinite time, as can be seen in Eq. (32). Thus, c_3 in Eq. (31) is found to be zero. Hence,

$$V = c_4 \exp\left(-\tau\sqrt{0.25 - \lambda_n^2}\right) \tag{33}$$

or

$$u = \sum_0^\infty c_n J_0(\lambda_n X) \exp\left(\frac{-\tau}{2} - \tau\sqrt{0.25 - \lambda_n^2}\right) \tag{34}$$

The c_n can be solved for from the initial condition by using the principle of orthogonality for Bessel functions. At time 0, the LHS And RHS are multiplied by $J_0(\lambda_m X)$. Integration between the limits of 0 and R is performed. When n is not m, the integral is zero from the principle of orthogonality. Thus, when $n = m$,

$$c_n = \frac{-\int_0^R J_0(\lambda_n X)}{\int_0^R J_0^2(c\lambda_n X)} \tag{35}$$

It can be noted from Eq. (34) that when

$$\tfrac{1}{4} < \lambda_n^2 \tag{36}$$

the solution will be periodic with respect to the time domain. This can be obtained by using de Moivre's theorem and obtaining the real part to $\exp(-i\tau\sqrt{\lambda_n^2 - 0.25})$. Thus, for materials with relaxation times greater than a certain limiting value, the solution for temperature will exhibit subcritical damped oscillations. Hence,

$$\tau_r > \frac{R^2}{9.62\alpha} \tag{37}$$

Thus, a bifurcated solution is obtained. From Eq. (34) it can also be seen that all terms in the infinite series will be periodic [i.e., even for $n = 2$ when Eq. (37) is valid]:

$$u = \sum_0^\infty c_n J_0(\lambda_n X) \cos\left(\tau\sqrt{\lambda_n^2 - 0.25}\right) \tag{38}$$

Thus, the transient-temperature profile in a cylinder is obtained for a step change in temperature at the surface of the cylinder using the modified Fourier's heat conduction law. For materials with relaxation times greater than $R^2/9.62\alpha$ (where R is the radius of the cylinder), subcritical damped oscillations can be seen in the transient-temperature profile. The exact solution for a finite cylinder subject to constant wall temperature using finite-speed heat conduction is derived by the method of separation of variables. It is a bifurcated solution. For certain values of λ, the time portion of the solution is cosinous and damped and for others it is an infinite series of the Bessel function of the first kind and one-half order and decaying exponential in time. It can also be shown that for terms in the infinite series with n greater than 2, the contribution to the solution will be periodic for small R.

IV. Conclusions

The main conclusions from the study are as follows:

1) Use of the final condition in time leads to bounded solutions.

2) The temperature overshoot problem can be attributed to the use of a physically unrealistic time condition.

3) The analytical solution obtained for the finite sphere and finite cylinder is found to be bifurcated.

4) For materials with large values of relaxation times, such as those given in Eqs. (21) and (27), subcritical damped oscillations in temperature can be found.

5) The exact solution for the finite cylinder and sphere does not violate the second law of thermodynamics.

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